Composite Fractional Power Wavelets

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ABSTRACT

Wavelets have a tremendous ability to extract signals from noisy environments. However, the use of wavelets can be computationally expensive. The number of computations increases with the size of the wavelet family. Here a wavelet family is combined into a single complex-valued signal that can then be used to extract information from an input signal. The advantage is that the expense of computation is that of a single correlation rather than the several correlations required by the wavelet family. This new filter is constructed using a phase-encoded fractional power filter and offers the user the option of manipulating the trade-off between generalization and discrimination that is inherent in first-order filtering. The result is a computationally cheaper method of using wavelets to detect signals embedded in noise.

1. INTRODUCTION

The ability of wavelets to extract important information from signals has long been known. The multi-resolution ability of this data analysis is far more powerful than traditional frequency analysis. The unfortunate aspect of wavelets is that there can be many wavelets for a single signal. Analysis of a signal using wavelets can therefore involve a large number of computations. One such example is the 'Cocktail Party' problem which used a bank of wavelets to identify a signal embedded in noise.[1]

However, this method required that each signal have its own set of wavelets. This is computationally expensive. This paper will use composite filtering to combine the set of wavelets from different training signals to create a wavelet filter that can extract a variety of signals embedded in noise.

Composite filters are filters trained from a weighted linear combination of signals. The one used here is the Fractional Power Filter (FPF).[2] This filter has the ability to train on several filters and manipulate the trade-off between generalization and discrimination. The FPF provides the ability to make a single wavelet filter that can recognize all of the training inputs and demonstrates successful recognition when the input is in the presence of strong random noise.

2. FOUNDATIONS

The two main foundations that are used here are wavelets and the FPF. This section provides a quick review of these two sciences.

2.1. Wavelets

The use of wavelets is a method that can extract from a signal multi-resolution frequency information. In that respect it is more powerful than Fourier analysis. There are a variety of wavelets that can be used, but for the purpose here only the Haar wavelet (which is the simplest) will be employed.

A signal, f(x), is decomposed by a family of wavelet functions by,

$$f(x) = c_{00}\phi(x) + \sum_{j=0}^{n-1} \sum_{k=0}^{2^j-1} d_{jk}\psi_{jk}(x) \qquad (1)$$

where $\psi_{jk}(x)$ is a set of orthonormal wavelet functions, d_{jk} are the coefficients for these functions, $\phi(x)$ is a unit function, c_{00} is its respective coefficient.

The Haar function is quite simple is 1 for $0 \le x < 0.5$ and -1 for $0.5 \le x < 1.0$, and the family of wavelets $\psi_{jk}(x)$ are computed by

$$\psi_{jk}(x) = \sqrt{2^j}\,\psi\left(2^j x - k\right) \tag{2}$$

where ψ is the original Haar wavelet. The coefficients d_{jk} are calculated by $\mathbf{M}^{-1}$ f, where the columns of **M** are **1** and ψ_{jk} in order.

The signal, **f**, can thus be decomposed and reconstructed through these wavelets. Of course, much work has been done with wavelets to improve the computations required and apply them to specific problems. Furthermore, the Haar wavelet is only one of many choices.

2.2. The Fractional Power Filter

The fractional power filter (FPF) is a composite filter that enables the user manipulate the trade-off between generalization and discrimination that is inherent in first-order filters. The calculation of this filter begins with a training vector set **X**, which consists of N vectors $\mathbf{x}_n$, n=1..N. The user associates each training vector with a constraint value. The constraint value can be any scalar and will be the result of the inner product between the filter and respective training vectors. These values are placed into a constraint vector, **c**, such that c_n is associated with $\mathbf{x}_n$, for all n.

The filter is a weighted linear combination of the training inputs, $\mathbf{h} = \sum_n \alpha_n \mathbf{x}_n$. Usually, the length of the training vectors is far greater than the number of training vectors. Thus, to determine **h** the psuedo-inverse is employed,

$$\vec{\mathbf{h}} = \mathbf{D}^{-1}\mathbf{X}\left[\mathbf{X}^T\mathbf{D}^{-1}\mathbf{X}\right]^{-1}\vec{c} \tag{3}$$

where the columns of **X** are the training vectors and

$$D_{ij} = \frac{\delta_{ij}}{N}\sum_n |X_{n,i}|^p \tag{4}$$

The power term, p, is a scalar ranging from 0 to 2. When p=0, **D** becomes **1** and **h** becomes a simple weighted linear combination. When p>0, this system will enhance the weaker components of all **x**'s. Thus, when the **x**'s are the Fourier components of inputs, and these inputs are slowly varying, then values of p>0 will tend to edge enhance. Since edges are important for discrimination and less so for generalization this allows the user to manipulate a trade-off between the two by changing the value of p.

3. COMBINING WAVELETS AND THE FPF

The composite wavelet is created by using the FPF to combine wavelet coefficients. This simple technique demonstrates efficient filtering and a robustness to strong noise.

3.1. An Example Problem.

In order to present the efficiency of the composite wavelet filter a sample problem is first established. The task defined by this problem is to recognize several signals embedded in a very noisy environment.

A set of N signals each of length D was created. A typical signal is shown in figure 1. These are designed to be near zero and the ends.

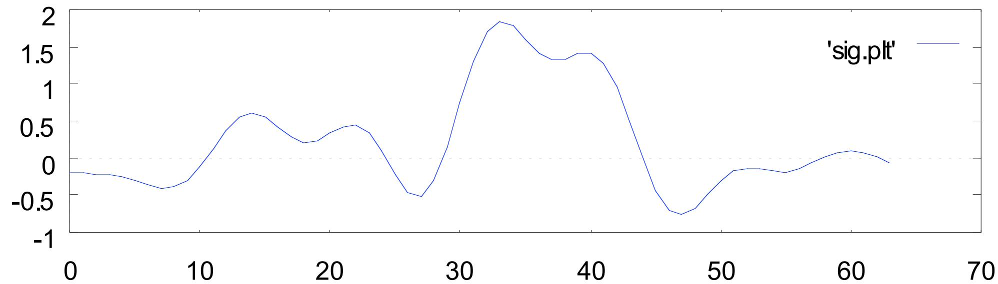

Figure 1. A Typical Signal.

The problem is to build a single wavelet set that will recognize all signals within the training set.

3.2. The Rejection of Traditional Methods

It is possible to build a wavelet filter[1] that recognizes specific signals. However, this would require that each signal have its wavelet filter. This becomes computationally expensive as N increases. Thus, neither wavelets nor the FPF are suited for the complete solution.

3.3. Composite Fractional Power Wavelets

To build the composite fractional power wavelet (CFPW) the wavelet coefficients for each signal are placed in the columns of **X**. The FPF algorithm is used (p=0 works well) to create a single set of wavelet coefficients, **h**. A new signal, **f**, is reconstructed using equation 1. Thi is the CFPW and is used to compare to the original training inputs.

The first test is that of inner products. The inner product of all training inputs with the single signal **f** should be the constraint values associated with those training inputs. The total error for N=25 and D=64 was less than the machine precision. This precise recall is maintained as long as the condition number of the matrix $\mathbf{X}^T\mathbf{D}^{-1}\mathbf{X}$ remains reasonable.

While the inner product is a good indication that training has occurred, it is usually not good for determination of the presence of a target signal. For this operation the inputs are correlated with **f**. Of course, for the correlations with the training inputs the center value will be the inner product. However, it is desired that a correlation peak exist. If other points in the correlation are larger than the inner product value then it is difficult to claim that recognition has occurred. Furthermore, it is common that the location of the target signal within the data stream is not known and thus a correlation would be instrumental in the location of the target.

The ability of the system to create a significant correlation peak is dependent upon the matrix inversion in equation 3. When the condition of this number worsens so does the ability of this system to provide a spike. The condition number is linked to the similarity of the training vectors and the ratio D/N. N cannot be greater than D, and as D/N nears 1 the condition number tends to worsen.

The PCE[3] provides a measure of the quality of a correlation peak. It is measured by

$$PCE = \frac{C_{\square}}{\sqrt{\frac{1}{D}\sum_i C_i^2}} \qquad (5)$$

where C is the correlation and $C_{\square}$ is the value at the peak or a selected point. The chart in figure 2 plots the PCE' for the first trainer correlated with different **f**'s. The x-axis is the number of training vectors used in creating **f** scaled by (N=3+3*x), thus x=8 indicates that 27 trainers were used.

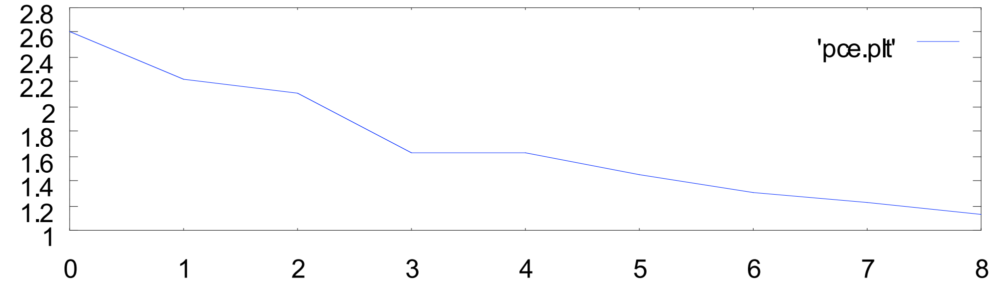

*Figure 2. The PCE' Plotted against the Number of Training Images, P, where P=(3+3*x).*

A typical correlation surface for a filter built from N=9 is shown in figure 3.

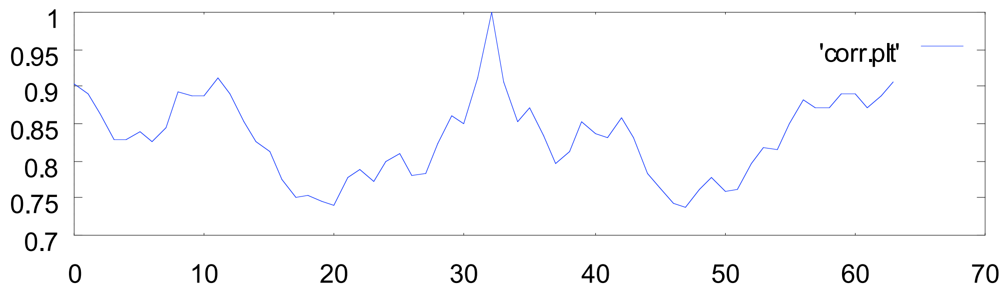

Figure 3. A Typical Correlation Surface for a Filter Built from 9 Training Inputs.

The correlation does peak at 1 at x=32, which is to be expected since that value is also the inner product. It would be desirable if the side lobes were smaller. In its current state this correlation is not useful since the side lobes are similar to the peak height. According to figure 2 this problem wrosens as N increases. This problem will be addressed below.

4. ANALYSIS OF FRACTIONAL POWER COMPOSITE WAVELETS

In this section several trials are run to extract the behavior of the composite fractional power wavelets (CFPW) under varying conditions.

4.1. Inner Products.

As seen the inner product of the training images with the CFPW is exactly the constraint value (within the computer precision). This relationship begins to fail when the condition number of the matrix $X^T D^{-1} X$ becomes unreasonable. The condition that causes this is when one of the trainers becomes a linear combination of the others. This will occur when the number of trainers exceeds the number of elements in those trainers.

However, the use of inner products to detect a target signal is not very useful. This would required that the user know the location of the target signal, and often the user is required to find the location of the target.

4.2. Correlations

As seen in figure 3, the correlation of the CFPW with a training signal does provide a peak, but the side lobes are very high. In several trials the side lobes exceeded the value of the peak. This makes it difficult to find the target.

The constraint vector used to compute the CFPW (**c** in equation 3) can contain any complex value. So, far the trials have had all elements in **c** equal to 1. Commonly, the user would like to reject a signal. In other words, find certain targets and reject others. To accomplish this the respective value of **c** is set to 0.

For discussion a training signal that has a constraint of 0 is denoted by T^0 and a training signal that has a constraint of 1 is denoted T^1. Figures 4 displays the correlations of (T^0,CFPW) and (T^1,CFPW). It can be seen that the side lobe activity, even though large in value, is very similar for these cases of opposing class. At the center of the correlation surface the two do differ by the constraint values.

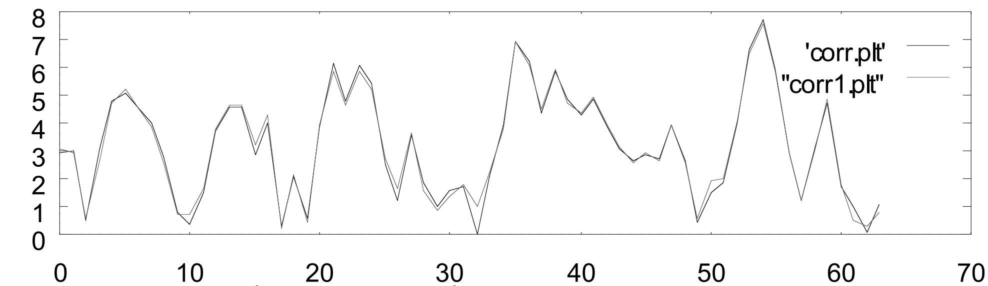

Figure 4. Correlations with a T^0 ('corr.plt') and a T^1 ('corr1.plt')

Thus, to find a target correlations the correlation of a T^0 is subtracted to eliminate the side lobe activity. The result of this subtraction is shown in figure 5.

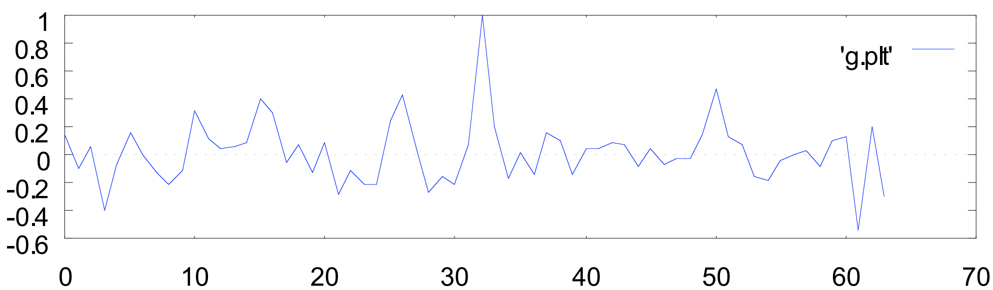

Figure 5. The subtraction of the two correlation surfaces in figure 4.

Correlations of the CFPW are now quite possible if the correlations are reduced by a T^0 correlation.

4.3. Noise Test

Next, the inner products of the CFPW with both a T^0 and a T^1 is considered. The noise was randomly added to the input. In figures 6 and 7 the results are displayed. The x-axis is the percent of noise when compared to the maximum value of the clean signal. So, x=0.3 indicates that the additive noise is 30% of the maximum value of the signal. The error bars indicate 1 sigma of the standard deviation of 100 trials. The dot in the middle of each error bar is the average value for those 100 trials.

This chart has two different sets of trials. One is for T^0 (zero.plt) and the other is for T^1 (one.plt). At x=0 there is no noise and so all of the inner products of T^0 are 0 and likewise all of the inner products of T^1 are 1.

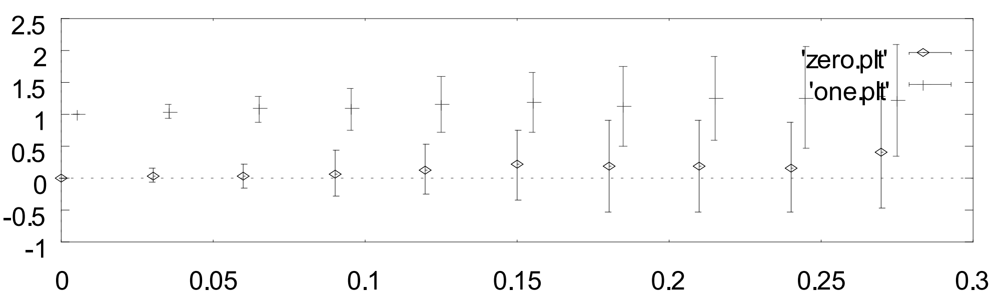

Figure 6. The noise test for N=9.

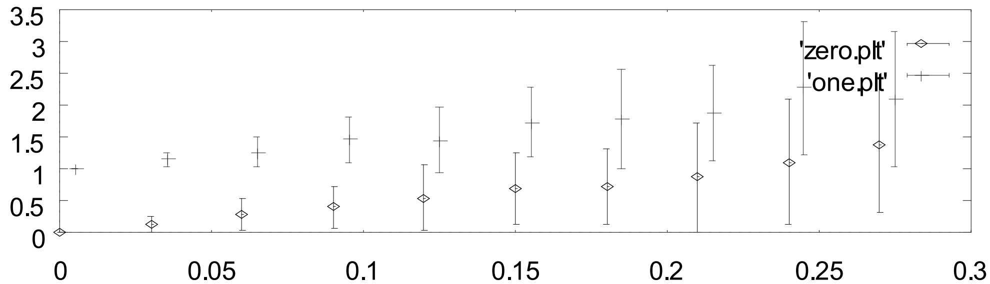

Figure 7. The noise test for N=25.

Obviously, at about 12% noise discrimination between the classes will begin to see some error because the error bars now overlap.

4.4. Warp Test

The warp test measures the performance when the original signal is squeezed. The x-axis measures the amount that the signal is shrunk in the x-axis by 1-0.05*x. Thus, when x=5, the new signal is 75% of the original length.

Figure 8 shows the results of this test. The trials for T^0 are charted as ‘zerow.plt’ and the trials for T^1 are charted as ‘onew.plt’. The conclusion is that when the signal is more than 90% of its original size discrimination between the two classes is possible.

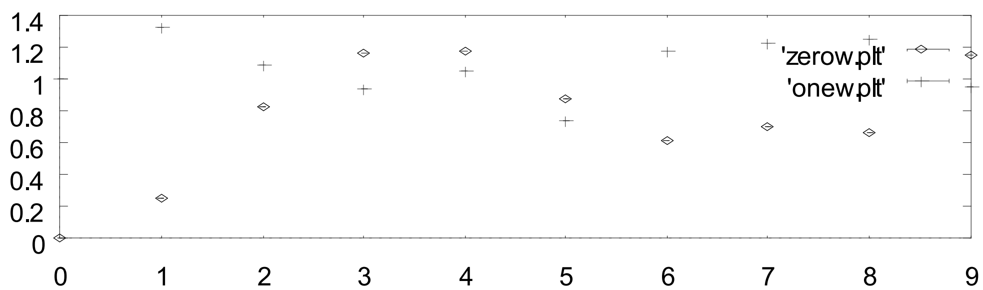

Figure 8. The warp test.

4.5. The Effects of the Fractional Power

The use of the FPF algorithm allows for the option of altering the behavior of the construction of the CFPW. The previous tests all had the fractional power set to 0. This is equivalent to weighted linear combinations.

The range of the fractional power is [0,2]. In the practice of traditional filtering it has been noted that a fractional power of 1.2 and above provides results very similar to a fractional power of 2.0. The effect of the fractional power is not linear.

The correlation results and the warp test results showed no appreciable difference. However, the noise tests were drastically different. Figure 9 displays the results for N=25, D=64, and the fractional power set to 0.9.

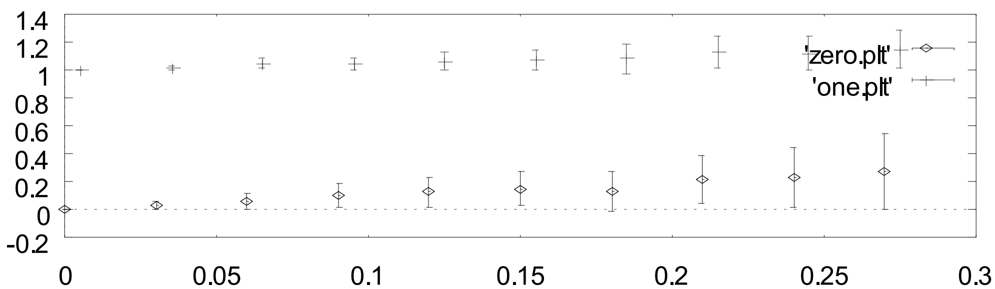

Figure 9. The noise test for p=0.9.

This test which used a total of 2000 trials indicates that the system is far more robust to noise when the fractional power is raised. Interestingly, this is the opposite effect experienced when the FPF is used for filtering of signals (containing more energy in the lower frequencies than in the higher).

5. SUMMARY

This paper presents a method of constructing a composite fractional power wavelet (CFPW). This is a wavelet set composed from wavelet coefficietns from many training signals. A single signal can be created from the composite coefficients. This signal can then be used to identify all of the training signals. This paper explores the effect of embedding the training signals in noise and warping their x-axis in a linear fashion. This composite filter does show the ability to recognize signals under these stressed conditions.

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